A Study of Applications of I- Functions in A Slightly Different Type of Gamma Density Model

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Abstract: In the present paper, the author has studied about the structures which are the products and ratios of statistically independently distributed positive real scalar random variables. The author has derived the exact density of Generalized Gamma density by the Mellin Transform and Hankel Transform of the unknown density and after that the unknown density has been derived in terms of I-function by taking the inverse Mellin transform and Inverse Hankel Transform. A more general structure of generalized Gamma density has also discussed. Special cases in terms of H-function are also given.

Keywords: Generalized Gamma density, Wright's Generalized Hypergeometric Function, I-function, H-function, Mellin Transform, Inverse Mellin Transform, Hankel Transform, Inverse Hankel Transform.

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I. INTRODUCTION

Generalized Wright's Function ${}_2R_1(a,b;c,\omega;\mu;z)$ defined by Dotsenko [1, 2] has been denoted as

$$ {}_2R_1(a,b;c,\omega;\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k\frac{\omega}{\mu})}{\Gamma(c+k\frac{\omega}{\mu})} \frac{z^k}{k!} $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2\psi_1[z|^{(a,1),(b,\frac{\omega}{\mu})}_{(c,\frac{\omega}{\mu})}]. \quad (1.1) $$

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The H-function is defined by means od a Mellin-Barnes type integral in the following manner (Mathai and Saxena 1978)

$$H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}[z|_{(b_q,B_q)}^{(a_p,A_p)}] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q) \end{matrix} \right]$$

$$= \frac{1}{2\Pi i}\int_L \theta(s) z^{-s} ds, \qquad (1.2)$$

Where

$$\theta(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j + B_j s)\Pi_{j=1}^{n}\Gamma(1 - a_j - A_j s)}{\Pi_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\Pi_{j=n+1}^{p}\Gamma(a_j + A_j s)} \qquad (1.3)$$

The I-function introduced by Saxena [6] will be represented and defined in slightly different manner as follows:

$$I[x] = I_{p_i,q_i;r}^{m,n}[z] = I_{p_i,q_i;R}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i} \end{matrix} \right]$$

$$= \frac{1}{2\Pi i}\int_L \theta(s) z^{-s} ds, \qquad (1.4)$$

Where $i = (-1)^{\frac{1}{2}}$, $z \neq 0$ and $z^{-s} = exp[-sln|z| + iargz]$, where $|z|$ represents the natural logarithim of $|z|$.

Here

$$\theta(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j + \beta_j s)\Pi_{j=1}^{n}\Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^{R}\{\Pi_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - \beta_{ji} s)\Pi_{j=n+1}^{p_i}\Gamma(a_{ji} + \alpha_{ji} s)\}} \qquad (1.5)$$

For $R = 1$, the I-function reduces to the H-function.

The Mellin transform of $f(x)$denoted by $Mf(x); s$ or $F(s)$ is given by

$$Mf(x); s = \int_0^\infty x^{s-1} f(x) dx \qquad (1.6)$$

The Hankel transform of $f(x)$denoted by $H_\nu f(x); p$ or $F_\nu(p)$ is defined as

$$H_\nu f(x); p = \int_0^\infty x J_\nu(px) f(x) dx \qquad (1.7)$$

II. GENERAL STRUCTURES

A real scalar random variable x is said to have a generalized gamma density when the density is of the form

$$f(x) = \begin{cases} \frac{\beta A^{\frac{\alpha+tm}{\beta}}}{{}_2R_1(a,b;c,\omega;\mu;p)\Gamma(\frac{\alpha+tm}{\beta})} x^{\alpha-1} e^{-ax^{\beta}_2 R_1(a,b;c,\omega;\mu;px^t)}, & x > 0, A > 0, \alpha > 0, \beta > 0 \\ 0, & elsewhere \end{cases} \tag{2.1}$$

Let E denote the mathematical expectation,the h^{th} moment of x, when x has the density in (2.1), is given by

$$E(x^h) = \frac{1}{A^{\frac{h}{\beta}}} \frac{\Gamma(\frac{\alpha+tm+h}{\beta})}{\Gamma(\frac{\alpha+tm}{\beta})} \tag{2.2}$$

For $Re(\alpha + tm + h) > 0$

Usually, in statistical problems, the parameters are real; hence, we will assume that the parameters a, α and β are real.Let

$$u = x_1 x_2 .... x_k \tag{2.3}$$

Where x_j has the density in (2.1) with the parameters $a_j > 0, \alpha_j > 0, \beta_j > 0, j = 1, 2, ..., k$ and let $x_1, ..., x_k$ be statistically independently distributed. In the standard terminology in statistical literature,the h^{th} moment of u, when u has the density in (2.1), is given by

$$E(u^h) = [E(x_1^h)][E(x_2^h)]...[E(x_k^h)]$$

$$= \Pi_{j=1}^{k} \frac{1}{A_j^{\frac{h}{\beta_j}}} \frac{\Gamma(\frac{\alpha_j+tm+h}{\beta_j})}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} \tag{2.4}$$

For $Re(\alpha_j + tm + h) > 0; j = 1, 2, ..., k$

Consider a set of real scalar random variables $x_1, ..., x_k$ mutually independently distributed, where s_j has the density in (2.4),(2.5) and (2.6)respectively with the parameters $(a_j, \gamma_j, \eta_j, \delta_j); j = 1, ..., k$ and consider the product$u = x_1 ... x_k$

Then the Mellin Transform of $g(u)$ is obtained from the property of the stastical independent and are given by

$$M[u^{s-1}] = \Pi_{j=1}^{k} \frac{A^{\frac{1}{\beta_j}} \Gamma(\frac{\alpha_j+tm-1}{\beta_j} + \frac{s}{\beta_j})}{A^{\frac{s}{\beta_j}} \Gamma(\frac{\alpha_j+tm}{\beta_j})} \tag{2.5}$$

For $Re(\alpha_j + tm + s - 1) > 0$.

The unknown density $g(u)$ of u is available in terms of I-function by the inverse Mellin transform of (2.5), that is,

$$g(u) = \begin{cases} \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} I_{0,k}^{k,0} [\Pi_{j=1}^{k} A_j^{\frac{1}{\beta_j}} u |_{(\frac{\alpha_j+tm-1}{\beta_j}, \frac{1}{\beta_j});j=1,\ldots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \tag{2.6}$$

The Hankel transform of $g(u)$ is obtained from the property of the stastical independent and are given by

$$H[uJ_\nu(pu)] = H[x_1 J_\nu(px_1)] H[x_2 J_\nu(px_2)] \ldots H[x_k J_\nu(px_k)]$$

$$= J_\nu(p) \Pi_{j=1}^{k} \frac{A^{\frac{1}{\beta_j}} \Gamma(\frac{\alpha_j+tm-1}{\beta_j} + \frac{s}{\beta_j})}{A^{\frac{s}{\beta_j}} \Gamma(\frac{\alpha_j+tm}{\beta_j})} \tag{2.7}$$

For $Re(\alpha_j + tm + s - 1) > 0$.

The unknown density $g(u)$ of u is available from the inverse Hankel transform of (2.7), that is,

$$g(u) = \begin{cases} J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} I_{0,k}^{k,0} [\Pi_{j=1}^{k} A_j^{\frac{1}{\beta_j}} |_{(\frac{\alpha_j+tm-1}{\beta_j}, \frac{1}{\beta_j});j=1,\ldots,k}^{-------}], 0 < u < \infty \\ \\ 0, elsewhere \end{cases} \tag{2.8}$$

If we consider more general structures in the same category. For example, consider the structure

$$u_1 = x_1^{\gamma_1} \ldots x_k^{\gamma_k}; \gamma_k > 0, j = 1, \ldots, k \tag{2.9}$$

Where $x_{1,\ldots,} x_k$ mutually independently distributed as in (2.3). Then, the Mellin transform of $g(u_1)$ is given as

$$M[u_1^{s-1}] = \Pi_{j=1}^{k} \frac{A^{\frac{\gamma_j}{\beta_j}} \Gamma(\frac{\alpha_j+tm-\gamma_j}{\beta_j} + \frac{s\gamma_j}{\beta_j})}{A^{\frac{s\gamma_j}{\beta_j}} \Gamma(\frac{\alpha_j+tm}{\beta_j})} \tag{2.10}$$

For $Re(\alpha_j + tm + s - 1) > 0, \gamma_j > 0$.

The unknown density $g(u_1)$ of u_1 is available from the inverse Mellin transform of (2.10), that is,

$$g(u_1) = \begin{cases} \Pi_{j=1}^{k} \frac{A_j^{\frac{\gamma_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} I_{0,k}^{k,0}[\Pi_{j=1}^{k} A_j^{\frac{\gamma_j}{\beta_j}} u_1 |_{(\frac{\alpha_j+tm-\gamma_j}{\beta_j}, \frac{\gamma_j}{\beta_j});j=1,\dots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \tag{2.11}$$

The Hankel transform of $g(u_1)$ is obtained from the property of the stastical independent and are given by

$$H[u_1 J_\nu(pu_1)] = H[x_1^{\gamma_1} J_\nu(px_1^{\gamma_1})] H[x_2^{\gamma_2} J_\nu(px_2^{\gamma_2})] \dots H[x_k^{\gamma_k} J_\nu(px_k^{\gamma_k})]$$

$$= J_\nu(p) \Pi_{j=1}^{k} \frac{A^{\frac{\gamma_j}{\beta_j}} \Gamma(\frac{\alpha_j+tm-\gamma_j}{\beta_j} + \frac{s\gamma_j}{\beta_j})}{A^{\frac{s\gamma_j}{\beta_j}} \Gamma(\frac{\alpha_j+tm}{\beta_j})} \tag{2.12}$$

For $Re(\alpha_j + tm + s - 1) > 0, \gamma_j > 0, s = \nu + 2r + 2 > 0$.

The unknown density $g(u_1)$ of u_1 is available from the inverse Hankel transform of (2.12), that is,

$$g(u_1) = \begin{cases} J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{\gamma_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} I_{0,k}^{k,0}[\Pi_{j=1}^{k} A_j^{\frac{\gamma_j}{\beta_j}} |_{(\frac{\alpha_j+tm-\gamma_j}{\beta_j}, \frac{\gamma_j}{\beta_j});j=1,\dots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \tag{2.13}$$

Where $Re(\alpha_j + tm + s - 1) > 0, \gamma_j > 0, s = \nu + 2r + 2 > 0$.

A More General Structure We can consider more general structures. Let

$$w = \frac{x_1, \dots, x_r}{x_{r+1}, \dots, x_k} \tag{2.14}$$

Where $x_1, \dots, x_k$ are mutually independently distributed real random variables having the density in (2.1) with x_j having parameters $\alpha_j, \beta_j; j = 1, \dots, k$, Then the Mellin transform of the density $g(w)$ is given as,

$$M[w^{s-1}] = M[x_1^{s-1}] \dots M[x_r^{s-1}] M[x_{r+1}^{-(s-1)}] \dots M[x_k^{-(s-1)}] \tag{2.15}$$

$$= \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} \{ \frac{\Gamma(\frac{\alpha_j+tm-1}{\beta_j} + \frac{s}{\beta_j})}{A_j^{\frac{s}{\beta_j}}} \} \{ \frac{\Gamma(\frac{\alpha_j+tm+1}{\beta_j} - \frac{s}{\beta_j})}{A_j^{\frac{-s}{\beta_j}}} \} \tag{2.16}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0$.

The unknown density $g(w)$is obtained by taking the inverse Mellin transform of (2.16).That is

$$g(w) = \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} H_{0,r}^{r,0} [\Pi_{j=1}^{r} A_j^{\frac{1}{\beta_j}} w |_{(\frac{\alpha_j+tm-1}{\beta_j}, \frac{1}{\beta_j}); j=1,\dots,r}^{------}]$$

$$I_{k-r,0}^{0,k-r} [\Pi_{j=r+1}^{k} A_j^{-\frac{1}{\beta_j}} w |_{----}^{(1-\frac{\alpha_j+tm+1}{\beta_j}, \frac{1}{\beta_j}); j=r+1,\dots,k}] \tag{2.17}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0$.

The Hankel transform of the density $g(w)$ of w is given as

$$H[wJ_\nu(pw)] = H[x_1 J_\nu(px_1)] H[x_2 J_\nu(px_2)] \dots H[x_r J_\nu(px_r)]$$

$$H[x_{r+1}^{-1} J_\nu(px_{r+1}^{-1})] \dots H[x_k^{-1} J_\nu(px_k^{-1})] \tag{2.18}$$

$$= J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} \left\{ \frac{\Gamma(\frac{\alpha_j+tm-1}{\beta_j} + \frac{s}{\beta_j})}{A_j^{\frac{s}{\beta_j}}} \right\} \left\{ \frac{\Gamma(\frac{\alpha_j+tm+1}{\beta_j} - \frac{s}{\beta_j})}{A_j^{\frac{-s}{\beta_j}}} \right\} \tag{2.19}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0$.

The unknown density $g(w)$is obtained by taking the inverse Hankel transform of (2.19).That is

$$g(w) = J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} H_{0,r}^{r,0} [\Pi_{j=1}^{r} A_j^{\frac{1}{\beta_j}} w |_{(\frac{\alpha_j+tm-1}{\beta_j}, \frac{1}{\beta_j}); j=1,\dots,r}^{------}]$$

$$I_{k-r,0}^{0,k-r} [\Pi_{j=r+1}^{k} A_j^{-\frac{1}{\beta_j}} w |_{----}^{(1-\frac{\alpha_j+tm+1}{\beta_j}, \frac{1}{\beta_j}); j=r+1,\dots,k}] \tag{2.20}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0$.

Now, we consider more general structures in the same category. For example, consider the structure

$$w_1 = \frac{x_1^{\delta_1} \dots x_r^{\delta_r}}{x_{r+1}^{\delta_{r+1}} \dots x_k^{\delta_k}} \tag{2.21}$$

Where $x_1, \dots, x_k$ are mutually independently distributed real random variables having the density in (2.1) with x_j having parameters $\alpha_j, \beta_j; j = 1, \dots, k$, Then the Mellin transform of the density $g(w_1)$ is given as:

$$M[w_1^{s-1}] = M[x_1^{\delta_1(s-1)}] \dots M[x_r^{\delta_r(s-1)}] M[x_{r+1}^{-\delta_r(s-1)}] \dots M[x_k^{-\delta_k(s-1)}] \tag{2.22}$$

$$= \Pi_{j=1}^{k} \frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} \left\{ \frac{\Gamma(\frac{\alpha_j+tm-\delta_j}{\beta_j} + \frac{s\delta_j}{\beta_j})}{A_j^{\frac{s\delta_j}{\beta_j}}} \right\} \left\{ \frac{\Gamma(\frac{\alpha_j+tm+\delta_j}{\beta_j} - \frac{s\delta_j}{\beta_j})}{A_j^{\frac{-s\delta_j}{\beta_j}}} \right\} \tag{2.23}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

The unknown density $g(w_1)$is obtained by taking the inverse Mellin transform of (2.23).That is

$$g(w_1) = \Pi_{j=1}^{k} \frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} I_{0,r}^{r,0} \left[\Pi_{j=1}^{r} A_j^{\frac{\delta_j}{\beta_j}} w_1 \Big|_{(\frac{\alpha_j+tm-\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=1,\dots,r}^{------} \right]$$

$$\times I_{k-r,0}^{0.k-r} \left[\Pi_{j=r+1}^{k} A_j^{-\frac{\delta_j}{\beta_j}} w_1 \Big|_{----}^{(1-\frac{\alpha_j+tm+\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=r+1,\dots,k} \right] \tag{2.24}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

The Hankel transform of the density $g(w_1)$ of w_1 is given as

$$H[w_1 J_\nu(pw)] = H[x_{\delta_1} J_\nu(px_1^{\delta_1})] H[x_2^{\delta_2} J_\nu(px_2^{\delta_2})] \dots H[x_r^{\delta_r} J_\nu(px_r^{\delta_r})]$$

$$\times H[x_{r+1}^{-\delta_{r+1}} J_\nu(px_{r+1}^{-\delta_{r+1}})] \dots H[x_k^{-\delta_k} J_\nu(px_k^{-\delta_k})] \tag{2.25}$$

$$= J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} \left\{ \frac{\Gamma(\frac{\alpha_j+tm-\delta_j}{\beta_j} + \frac{s\delta_j}{\beta_j})}{A_j^{\frac{s\delta_j}{\beta_j}}} \right\} \left\{ \frac{\Gamma(\frac{\alpha_j+tm+\delta_j}{\beta_j} - \frac{s\delta_j}{\beta_j})}{A_j^{\frac{-s\delta_j}{\beta_j}}} \right\} \tag{2.26}$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

The unknown density $g(w_1)$is obtained by taking the inverse Hankel transform of (2.26).That is

$$g(w_1) = J_\nu(p)\Pi_{j=1}^{k}\frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}I_{0,r}^{r,0}\left[\Pi_{j=1}^{r}A_j^{\frac{\delta_j}{\beta_j}}w_1|_{(\frac{\alpha_j+tm-\delta_j}{\beta_j},\frac{\delta_j}{\beta_j});j=1,\ldots,r}^{------}\right]$$

$$\times I_{k-r,0}^{0,k-r}\left[\Pi_{j=r+1}^{k}A_j^{-\frac{\delta_j}{\beta_j}}w_1|_{----}^{(1-\frac{\alpha_j+tm+\delta_j}{\beta_j},\frac{\delta_j}{\beta_j});j=r+1,\ldots,k}\right] \quad (2.27)$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

III. SPECIAL CASES:

for setting $R = 1$ in (2.6),the unknown density $g(u)$ of u is available in terms of H-function, that is,

$$g(u) = \begin{cases} \Pi_{j=1}^{k}\frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,k}^{k,0}[\Pi_{j=1}^{k}A_j^{\frac{1}{\beta_j}}u|_{(\frac{\alpha_j+tm-1}{\beta_j},\frac{1}{\beta_j});j=1,\ldots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \quad (3.1)$$

For $\beta_j = 1, j = 1, \ldots, k$, the H-function reduces to the G-function.

On taking $R = 1$ in (2.8)The unknown density $g(u)$ of u is available in terms of H-function, that is,

$$g(u) = \begin{cases} J_\nu(p)\Pi_{j=1}^{k}\frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,k}^{k,0}[\Pi_{j=1}^{k}A_j^{\frac{1}{\beta_j}}|_{(\frac{\alpha_j+tm-1}{\beta_j},\frac{1}{\beta_j});j=1,\ldots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \quad (3.2)$$

For $\beta_j = 1, j = 1, \ldots, k$, the H-function reduces to the G-function.

For $R = 1$ in (2.11)The unknown density $g(u_1)$ of u_1 is available in terms of H-function, that is,

$$g(u_1) = \begin{cases} \Pi_{j=1}^{k}\frac{A_j^{\frac{\gamma_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,k}^{k,0}[\Pi_{j=1}^{k}A_j^{\frac{\gamma_j}{\beta_j}}u_1|_{(\frac{\alpha_j+tm-\gamma_j}{\beta_j},\frac{\gamma_j}{\beta_j});j=1,\ldots,k}^{-------}], 0 < u < \infty \\ 0, elsewhere \end{cases} \quad (3.3)$$

If we put $R=1$ in (2.13),The unknown density $g(u_1)$ of u_1 is available in terms of H-function, that is,

$$g(u_1)=\begin{cases} J_\nu(p)\Pi_{j=1}^{k}\frac{A_j^{\frac{\gamma_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,k}^{k,0}[\Pi_{j=1}^{k}A_j^{\frac{\gamma_j}{\beta_j}}|_{(\frac{\alpha_j+tm-\gamma_j}{\beta_j},\frac{\gamma_j}{\beta_j});j=1,\dots,k}^{-------}],0<u<\infty \\ 0, elsewhere \end{cases} \tag{3.4}$$

Where $Re(\alpha_j+tm+s-1)>0,\gamma_j>0,s=\nu+2r+2>0.$

For $\beta_j=1=\gamma_j,j=1,...,k$, the H-function reduces to the G-function.

If we put $R=1$ in (2.17),The unknown density $g(w)$is obtained in terms of H-function.That is

$$g(w)=\Pi_{j=1}^{k}\frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,r}^{r,0}\left[\Pi_{j=1}^{r}A_j^{\frac{1}{\beta_j}}w|_{(\frac{\alpha_j+tm-1}{\beta_j},\frac{1}{\beta_j});j=1,\dots,r}^{------}\right]$$
$$\times H_{k-r,0}^{0,k-r}\left[\Pi_{j=r+1}^{k}A_j^{-\frac{1}{\beta_j}}w|_{----}^{(1-\frac{\alpha_j+tm+1}{\beta_j},\frac{1}{\beta_j});j=r+1,\dots,k}\right] \tag{3.5}$$

For $Re(\alpha_j+tm\pm(s-1))>0.$

For $\beta_j=1,j=1,...,k$, the H-function reduces to the G-function.

If we put $R=1$ in (2.20),The unknown density $g(w)$is obtained in terms of H-function.That is

$$g(w)=J_\nu(p)\Pi_{j=1}^{k}\frac{A_j^{\frac{1}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})}H_{0,r}^{r,0}\left[\Pi_{j=1}^{r}A_j^{\frac{1}{\beta_j}}w|_{(\frac{\alpha_j+tm-1}{\beta_j},\frac{1}{\beta_j});j=1,\dots,r}^{------}\right]$$
$$\times H_{k-r,0}^{0,k-r}\left[\Pi_{j=r+1}^{k}A_j^{-\frac{1}{\beta_j}}w|_{----}^{(1-\frac{\alpha_j+tm+1}{\beta_j},\frac{1}{\beta_j});j=r+1,\dots,k}\right] \tag{3.6}$$

For $Re(\alpha_j+tm\pm(s-1))>0.$

For $\beta_j=1,j=1,...,k$, the H-function reduces to the G-function.

Now, we consider more general structures in the same category. For example, consider the structure

If we put $R=1$ in (2.24),The unknown density $g(w_1)$is obtained in terms of H-function.That is

$$g(w_1) = \Pi_{j=1}^{k} \frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} H_{0,r}^{r,0} \left[\Pi_{j=1}^{r} A_j^{\frac{\delta_j}{\beta_j}} w_1 \Big|_{(\frac{\alpha_j+tm-\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=1,\ldots,r}^{------} \right]$$

$$\times H_{k-r,0}^{0,k-r} \left[\Pi_{j=r+1}^{k} A_j^{-\frac{\delta_j}{\beta_j}} w_1 \Big|_{----}^{(1-\frac{\alpha_j+tm+\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=r+1,\ldots,k} \right] \quad (3.7)$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

For $\beta_j = 1 = \delta_j, j = 1, \ldots, k$, the H-function reduces to the G-function.

If we put $R = 1$ in (2.27),The unknown density $g(w_1)$is obtained in terms of H-function.That is

$$g(w_1) = J_\nu(p) \Pi_{j=1}^{k} \frac{A_j^{\frac{\delta_j}{\beta_j}}}{\Gamma(\frac{\alpha_j+tm}{\beta_j})} H_{0,r}^{r,0} \left[\Pi_{j=1}^{r} A_j^{\frac{\delta_j}{\beta_j}} w_1 \Big|_{(\frac{\alpha_j+tm-\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=1,\ldots,r}^{------} \right]$$

$$\times H_{k-r,0}^{0,k-r} \left[\Pi_{j=r+1}^{k} A_j^{-\frac{\delta_j}{\beta_j}} w_1 \Big|_{----}^{(1-\frac{\alpha_j+tm+\delta_j}{\beta_j}, \frac{\delta_j}{\beta_j}); j=r+1,\ldots,k} \right] \quad (3.8)$$

For $Re(\alpha_j + tm \pm (s-1)) > 0, \delta_j > 0$.

For $\beta_j = 1 = \delta_j, j = 1, \ldots, k$, the H-function reduces to the G-function.

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